Senior Problem Set 1

Written by Andy Tran for the MaPS Correspondence Program

15 February 2021

Instructions

- Some (but not all) of the problems are based on the notes "*Bounding Arguments*".
- All problems are worth 7 points
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem, but you should attempt all of them
- Please submit your solutions to your mentor for marking and feedback.
- The due date for this problem set is **28 February, 2021, before 11.59pm**.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf document of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers on the Ed fourm but the solutions you submit must be written by yourself.

Problems

1. Let $p(x)$ be an integer polynomial such that $p(2)$ is divisible by 5 and $p(5)$ is divisible by 2. Prove that $p(7)$ is divisible by 10.
2. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers
$$a^2 + b + c$$
$$b^2 + c + a$$
$$c^2 + a + b$$
to be perfect squares.
3. In triangle ABC, let the tangent to its circumcircle at A intersect BC at point D. Prove that
$$AB^2 \cdot CD = AC^2 \cdot BD.$$
Recall that the circumcircle of a triangle is the circle passing through the three vertices.
4. Find all positive integers a, b, c, n such that
$$a! + b! + c! = 2^n$$
5. Imagine a 2020×2021 grid of unit screens, each of them can be either on or off. Initially, there are more than $2019 \cdot 2020$ unit screens which are on. In any 2×2 square of unit screens, as soon as there are 3 unit screens which are off, the fourth screen turns off automatically. Prove that the whole screen can never be totally off.